Translation nets: a survey

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1. Introduction

Let G be a group of order $s^2 > 1$ and let $\mathbb{H}$ be a set of subgroups of G satisfying:

(1.1) $|\mathbb{H}| = r$,

(1.2) $|U| = s$ for all U in $\mathbb{H}$,

(1.3) $UV = G$ for any two different elements U and V in $\mathbb{H}$.

Then $\mathbb{H}$ is called a *partial congruence partition in G with parameters s and r* (for short: (s, r)-*PCP*) while the elements of $\mathbb{H}$ are called *components*. For reasons of cardinality, (1.3) is equivalent to

(1.3′) $U \cap V = 1$ for any two different components U and V of $\mathbb{H}$.

We are interested in partial congruence partitions $\mathbb{H}$ in G because the incidence structure

(1.4) $N(\mathbb{H}) = (G, \{Ug \mid g \in G, U \in \mathbb{H}\}, \in)$

is a *net of order s and of degree r* (for short: an (s, r)-net), i.e., a finite, simple *affine* 1-*design,* additionally admitting G in a natural way as *translation group* acting regularly on the set of points of $N(\mathbb{H})$. Therefore $N(\mathbb{H})$ is called a *translation net of order s and of degree r.* (For geometric reasons, one usually

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assumes $r \geq 3$.) Observe that the parallel classes of $N(\mathbb{H})$ are exactly the sets of right cosets of the components of $\mathbb{H}$. Conversely, every translation net with translation group G can be coordinatized as in (1.4) for a suitable partial congruence partition $\mathbb{H}$ in G (see [33, 4]). Hence the existence of translation nets is a purely group-theoretic problem which can be formulated as follows:

(1.5) Let G be a group of order $s^2 > 1$. Determine the number

$$T(G) := \max\{r \leq s+1 \mid \text{there exists an } (s, r)\text{-PCP in } G\}$$

exactly or find at least bounds for it.

In this survey, we will discuss the existence problem for translation nets, the question of when a translation net is *maximal* and the *codes* of abelian translation nets. This updates the relevant section in Jungnickel [24].

2. Upper bounds for $T(G)$

It is easy to show that the degree r of an (s, r)-net N is at most $s+1$. Equality holds if and only if any two points are joined by exactly one line. In this case N is an *affine plane of order s*. Thus translation nets are generalizations of the well-known *translation planes* which were first studied by André [1]. The standard reference for translation planes is Lüneburg [27]. It is a fundamental fact that the translation group of a translation plane is elementary abelian. Therefore, we have:

(2.1) If G is an elementary abelian group of order $p^{2n} > 1$ (for some prime p), then $T(G) = p^n + 1$.

A natural question, which was first dealt with by Jungnickel [21] is to ask for upper bounds for the degree r of a translation net with translation group G provided that G is not elementary abelian. Theorem 2.2 is due to Frohardt [14] which is generalized by Jungnickel [23]. It says that upper bounds on $T(G)$ are found by studying the Sylow-subgroups of G.

Theorem 2.2. *Let G be a group of order $s^2 > 1$, p be a prime divisor of s and let P be a p-Sylow-subgroup of G. Then any (s, r)-PCP in G induces a partial congruence partition in P with the same degree r, i.e. $T(G) \leq T(P)$.*

This result is the main motivation for studying the existence of partial congruence partitions in p-groups. Theorem 2.3 solves problem (8.2.14) of Jungnickel [24].

Theorem 2.3 (Hachenberger [18]). *Let p be any prime and let G be a group of order p^{2n} which is not elementary abelian. If $n \geq 4$, then*

$$T(G) \leq (p^{n-1} - 1)(p-1)^{-1} = p^{n-2} + \cdots + p + 1.$$

For example, if G is a group of order 3^8 which is not elementary abelian, then $T(G) \leq 13$ while $T(\mathrm{EA}(3^8)) = 82$. Theorem 2.3 is a generalization of a result of Frohardt [14] where the particular case $p = 2$ is dealt with. As the cases $p = 2$ and p odd are not distinguished in the proof of Theorem 2.3, many of the arguments used there are different from Frohardt's, who used special facts about 2-groups.

3. Partial congruence partitions in groups of order p^4 and p^6

In this section we summarize some results on groups of order p^4 and p^6, two cases not covered by Theorem 2.3.

Theorem 3.1 (Hachenberger [17]). *Let G be a group of order p^4 satisfying $T(G) \geq 3$. Then one has one of the following cases:*

(3.1.1) *G is elementary abelian and $T(G) = p^2 + 1$.*

(3.1.2) *G is isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and $T(G) = p + 1$.*

(3.1.3) *p is odd and G is either isomorphic to $\langle x, y \mid x^{p^2} = y^{p^2} = 1,\ [x, y] = x^p \rangle$ or $E(p^3) \times \mathbb{Z}_p$. In both cases, one has $T(G) = p + 1$.*

(3.1.4) *$p = 2$ and G is either isomorphic to $\langle x, y \mid x^4 = y^4 = 1,\ [x, y] = x^2y^2, (x^2y^2)^2 = 1 \rangle$ or $D_4 \times \mathbb{Z}_2$. In both cases, one has $T(G) = 3$.*

(Here $\mathbb{Z}_a$ denotes the cyclic group of order a, $E(p^3)$ the extraspecial group of order p^3 and exponent p and D_4 the dihedral group of order 8.)

Theorem 3.2 (Hachenberger [18]). *Let G be a group of order p^6 satisfying $T(G) \geq p + 2$. Then one has one of the following cases:*

(3.2.1) *G is elementary abelian and $T(G) = p^3 + 1$.*

(3.2.2) *p is odd, G is isomorphic to $E(p^3) \times \mathrm{EA}(p^3)$ and $T(G) = p + 2$.*

(3.2.3) *G is isomorphic to $\langle a, b, c, u, v, x \mid$ all generators have order p; $[a, b] = u$, $[b, c] = v$, all further commutators in generators are equal to $1 \rangle$ and $T(G) = p + 2$.*

(3.2.4) *p is odd, G is isomorphic to the special group of exponent p with center of order p^3 and $T(G) = p^2 + 1$.*

The groups G of order 64 satisfying $T(G) \geq 4$ were already classified by Sprague [33] and Gluck [15]. In the main parts of the proof of Theorem 3.2 the cases $p = 2$ and p odd are not distinguished. The special interest of some authors in groups of even order stems from a theorem of Dillon [11] which states that $(2k, k)$-partitions in groups G of order $4k^2$ can be used to construct certain difference sets in G. This shows that partial congruence partitions also lead to other interesting objects studied in design theory (cf. Bailey and Jungnickel [3]). By results of Sprague [33], Gluck [15], Frohardt [14] and Hachenberger [17, 18] all groups G of order p^2k^2 (where $k \geq 3$ and p is the smallest prime divisor of $|G|$) satisfying $T(G) \geq k$ are classified.

4. Constructions and examples

In order to determine $T(G)$ exactly for some nontrivial group G of square order, one has to show the existence of a partial congruence partition of degree $T(G)$ in G. Therefore, examples of partial congruence partitions implicitly are needed in the proofs of Theorems 3.1 and 3.2. It is remarkable that in all but the groups in (3.2.3) any known example of a PCP of degree $T(G)$ contains at least one normal component. It seems therefore to be important to study such partial congruence partitions. This is done by Hachenberger and Jungnickel [19] and by Hachenberger [16] under a geometric and group theoretic point of view, respectively. A simple but important observation due to Sprague [33] shows that it is useful to divide the class of all partial congruence partitions into the following families:

(a) there exists no normal component,
(b) there exists exactly one normal component,
(c) there exist exactly two normal components,
(d) there exist at least three normal components (which already implies that the corresponding translation group is abelian).

A further fundamental result says that the existence of a partial congruence partition $\mathbb{H}$ with $|\mathbb{H}| \geq 2$ is equivalent to the existence of a so-called system of pairwise orthogonal othomorphisms between any two different components of $\mathbb{H}$.

Theorem and Definition 4.1 (Hachenberger and Jungnickel [19]). *Let $\mathbb{H}$ be an (s, r)-PCP in a group G of order $s^2 > 1$ and assume that $r \geq 2$. Let A and H be two different components of $\mathbb{H}$. For any U in $\mathbb{H} - \{A, H\}$ let γ_U be the following mapping:*

$$\gamma_U : H \to A, \qquad h \to \gamma_U(h), \qquad \textit{where } \{\gamma_U(h)^{-1}\} = A \cap Uh.$$

Then:

(4.1.1) *γ_U is a bijection.*
(4.1.2) $U = \{h\gamma_U(h) \mid h \in H\}$.
(4.1.3) *For any two different components U and V in $\mathbb{H} - \{A, H\}$ the mapping $\delta_{U,V} : H \to A$, $h \to \gamma_U(h)^{-1}\gamma_V(h)$ likewise is a bijection.*

Moreover, any partial congruence partition may be represented in this way. Any γ_U in $\Gamma(H, A) := \{\gamma_U \mid U \in \mathbb{H} - \{A, H\}\}$ is called an *(H, A)-orthomorphism* and $\Gamma(H, A)$ is called a *system of pairwise orthogonal (H, A)-orthomorphisms.*

If A is a normal subgroup of G in the situation above, then G is isomorphic to a semidirect product $S(H, A, \pi)$ for a suitable homomorphism $\pi : H \to \mathrm{Aut}(A)$. In this case, the mappings γ_U in $\Gamma(H, A)$ additionally satisfy the following property

$$(4.2) \qquad \gamma_U(h_1h_2) = \gamma_U(h_1)^{\pi(h_2)}\gamma_U(h_2) \quad \text{for all } h_1, h_2 \text{ in } H$$

and are therefore called *semi-isomorphisms of H onto A* (with respect to π), while $\Gamma(H, A)$ is called a *system of pairwise orthogonal semi-isomorphisms of H onto A* (with respect to π).

Example 4.3. Let p be an odd prime and G be isomorphic to $E(p^3) \times \mathrm{EA}(p^3)$, the group in (3.2.2), e.g.

$$G = \langle a, b, u, v, x, y \mid \text{all generators have order } p;\ [a, b] = y, \\ \text{all further commutators in generators are equal to } 1 \rangle .$$

Then $A := \langle av, b, y \rangle$ is a normal subgroup which is isomorphic to $E(p^3)$ while $H := \langle bx, v, uy \rangle$ is an elementary abelian subgroup of order p^3 which is a complement of A in G. For i in $\mathrm{GF}(p)$, the Galois field of order p, let

$$\varphi_i : \begin{cases} H \to A, \\ b^{\beta} u^{\gamma} v^{\alpha} x^{\beta} y^{\gamma} \to a^{-\alpha+i\gamma} b^{-\beta-i\alpha+i^2\gamma} v^{-\alpha+i\gamma} y^{-\gamma-\alpha\beta+i\beta\gamma+i\beta-i\binom{-\alpha+i\gamma}{2}}. \end{cases}$$

These mappings form a system of pairwise orthogonal semi-isomorphisms of H onto A with respect to conjugation of A by elements of H. The corresponding $(p^3, p+2)-\mathrm{PCP}$, $\mathbb{H} := \{A, H\} \cup \{\varphi_i(H) \mid i \in \mathrm{GF}(p)\}$, is of maximal possible degree by (3.2.2).

Partial congruence partitions in family (b) where the normal component A is abelian are of particular interest. Using some basic facts about the first cohomology group of the translation group G considered as an extension of A, one can actually construct such PCPs (see Hachenberger [16]). In this case an (H, A)-orthomorphism is a bijective *cocycle* of H onto A (i.e., a mapping satisfying (4.2) under the assumption that A is abelian). The set of all cocycles of H into A forms a subgroup with respect to pointwise multiplication which is isomorphic to the subgroup

$$U(H, A) := \{\varphi \in \mathrm{Aut}(G) \mid \varphi(a) = a \text{ for all } a \text{ in } A,\ \varphi(hA) = hA \text{ for all } h \text{ in } H\}$$

of $\mathrm{Aut}(G)$ (see Aschbacher [2, §17]). Therefore partial congruence partitions containing H and A can be described by suitable subsets of $U(H, A)$.

Example 4.4 (Hachenberger [16]). Let $\mathrm{GF}(q)$ be the Galois field of order q and odd characteristic p. Then

$$\begin{aligned} &(a_1, b_1, c_1, x_1, y_1, z_1)(a_2, b_2, c_2, x_2, y_2, z_2) \\ &\quad := (a_1 + a_2, b_1 + b_2, c_1 + c_2, x_1 + x_2 - a_2 b_1, \\ &\qquad y_1 + y_2 - a_2 c_1, z_1 + z_2 - b_2 c_1) \end{aligned}$$

defines a multiplication on the set G of 6-tupels over $\mathrm{GF}(q)$ which turns G into a non-abelian group of exponent p. The subgroups $H := \{(a, b, 0, x, 0, 0) \mid a, b, x \in \mathrm{GF}(q)\}$ and $A := \{(0, 0, c, 0, y, z) \mid c, y, z \in \mathrm{GF}(q)\}$ of G have order q^3 each and intersect trivially. A is an elementary abelian normal subgroup of G while H is a special group which is not normal in G. Let γ_1, η_1, ζ_1, γ_2, η_2, ζ_2 be any elements of GF(q). The mapping $\tau := \tau(\gamma_1, \eta_1, \zeta_1, \gamma_2, \eta_2, \zeta_2)$ defined by

$$\begin{aligned}(a, b, c, x, y, z)^\tau \\ := \Big(a,\ b,\ c + \gamma_1 a + \gamma_2 b,\ x,\ y + \eta_1 a + \eta_2 b - \gamma_1\binom{a}{2} + \gamma_2 x, \\ z + \zeta_1 a + \zeta_2 b - \gamma_2\binom{b}{2} - \gamma_1 x - \gamma_1 ab\Big)\end{aligned}$$

is an automorphism of G fixing the sets A and G/A elementwise and therefore by definition lies in $U(H, A)$. Now there exists an element $d = d(q)$ in GF(q) such that the polynomial $t^3 - t - d$ is irreducible in GF(q)[t]. Then the set

$$\Psi_d := \{\tau_{i,j} \mid i, j \in \mathrm{GF}(q)\} \quad \text{with } \tau_{i,j} := \tau(-j, 0, -2^{-1}(i+j), i, -2^{-1}dj, 0)$$

is a subgroup of order q^2 of $U(H, A)$ and the orbit of H under Ψ_d together with A forms a (q^3, q^2+1)-PCP in G.

Considering the case where H and A both are normal components in G, the orthomorphism method reduces to the well-known automorphism method (see [3]) which is based on ideas of Mann [28] for constructing certain sets of mutually orthogonal latin squares.

Automorphism method 4.5. Let K be a finite group. An automorphism τ of K is called fixed-point-free if $\tau = \mathrm{id}_K$ or if 1 is the only element of K fixed by τ. A subset Γ of Aut(K) is called a *system of pairwise orthogonal fixed-point-free automorphisms of K* if

(4.5.1) γ is fixed-point-free for all γ in Γ and

(4.5.2) $\gamma^{-1}\varphi$ is fixed-point-free for all φ, γ in Γ.

Let $f(K)$ denote the maximal cardinality of a set of pairwise orthogonal fixed-point-free automorphisms of K, and let $G := K \times K$, $A := \{(1, k) \mid k \in K\}$ and $H = \{(k, 1) \mid k \in K\}$. Then there exists an $(|K|, r)$-PCP in G containing the normal subgroups A and H as components if and only if $f(K) \geq r - 2$.

Example 4.6. Let p be a prime and let $n > 1$ be an integer which is not a power of 2. Furthermore, let $K = \mathrm{GF}(q)$ be the Galois field of order $q = p^n$ and let τ be an automorphism of K of odd order which is not the identity. Then

$$A(n, \tau) := \{u(a, b) \mid a, b \in K\}$$

with

$$u(a, b) := \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^\tau \\ 0 & 0 & 1 \end{pmatrix}$$

is a non-abelian group of order q^2 and

$$H := \{h(\lambda) : \lambda \in K^*\} \quad \text{with } h(\lambda) := \mathrm{diag}(1, \lambda, \lambda\lambda^\tau)$$

is a cyclic group of order $q-1$ acting in the following way as an automorphism group of $A(n, \tau)$:

$$u(a, b)^{h(\lambda)} := h(\lambda)^{-1} u(a, b) h(\lambda) = u(\lambda a, \lambda\lambda^\tau b).$$

It is not difficult to see that $h(\lambda)$ is fixed-point-free unless p is odd and $\lambda = -1$. Hence any complete system of representatives of the right cosets of the subgroup $\langle h(-1) \rangle$ in H forms a system of pairwise orthogonal fixed-point-free automorphisms of $A(n, \tau)$. As the order of $h(-1)$ is 2 if p is odd and 1 if $p = 2$, we therefore obtain

$$T(A(n, \tau) \times A(n, \tau)) \geq f(A(n, \tau)) + 2 \geq (q-1) \cdot \gcd(2, q-1)^{-1} + 2,$$

which proves the existence of a $(q^2, (q-1) \cdot \gcd(2, q-1)^{-1} + 2)$-PCP in the group $A(n, \tau) \times A(n, \tau)$.

We finally consider the abelian case. This was completely solved by Bailey and Jungnickel [3].

Theorem 4.7. *Let* $A = (\mathbb{Z}_p)^{m_1} \times \cdots \times (\mathbb{Z}_{p^a})^{m_a}$ *where* $m_a \neq 0$. *Then*

$$T(A \times A) = p^m + 1, \quad \textit{where } m = \min\{m_i \mid m_i \geq 1, i = 1, \ldots, a\}.$$

As any partial congruence partition in an abelian group can be constructed by Method 4.5, Theorem 4.7 together with an obvious direct product construction indeed completely solves the existence of translation nets with an abelian translation group (see [24]).

5. Maximal nets

In this section, we discuss the application of translation nets to a problem which has generated considerable interest ever since it was first considered by Bruck [6], i.e., the construction of *maximal* nets or, equivalently, *maximal sets of mutually orthogonal Latin squares* (MOLS). This is of particular interest to the question of the quality of the bound in Bruck's [7] celebrated completion theorem for nets. While this result roughly states that any net of order s and degree r is *imbeddable* into an affine plane of the same order provided that its *deficiency* $d = s + 1 - r$ (i.e., the number of 'missing' parallel classes) is only in the magnitude of $\sqrt[4]{s}$, the largest known examples of maximal nets all have deficiency about $\sqrt{s}$. The following recent result due to Metsch [29] considerably narrows this gap by lowering the bound required for imbeddability to about $\sqrt[3]{s}$.

Theorem 5.1. *Let N be an (s, r)-net of deficiency $d = s + 1 - r$ satisfying*

(5.1.1) $$3s > 8d^3 - 18d^2 + 8d + 4 - 2R(d^2 - d - 1) + 9R(R-1)(d-1)/2,$$

where $R \equiv d + 1 \pmod 3$ and $R \in \{0, 1, 2\}$. Then N can be imbedded into an affine plane of order s.

We recall that another result of Bruck [7] guarantees that an (s, r)-net N has at most one completion to an affine plane of order s provided that N has *small deficiency*, i.e., that the deficiency d of N satisfies

(5.1.2) $$s > (d-1)^2;$$

in this case, N has at most sd transversals, and equality holds if and only if N is imbeddable. These results are no longer true for nets with *critical deficiency*, i.e., with

(5.1.3) $$s = (d-1)^2,$$

as shown by Ostrom [32]. Here N has at most $2sd$ transversals, and equality holds if and only if there exist exactly two completions of N to an affine plane of order s.

In view of the preceding remarks, the construction of transversal-free nets of small or critical deficiency is of particular interest. The first examples of such nets were provided by Bruen [9, 10] who constructed transversal-free translation nets of order $s = p^2$ and deficiency $d = p$ (where p is an odd prime) and $d = p - 1$ (where $p \geq 5$ is a prime). We shall now sketch the known examples which can be obtained by means of translation nets. (For other recent constructions, see Evans [12, 13].) The remaining results of this section are due to Jungnickel [25] unless stated otherwise. As the following result shows, only elementary abelian translation groups have to be considered. (In fact, a more general result can be obtained, but the version given here suffices. The proofs use the bounds discussed in Section 2.)

Theorem 5.4. *Let N be an (s, r)-translation net with translation group G, where $s \neq 4$, let p be the smallest prime divisor of s, and assume that s is a square. If the deficiency d of N satisfies*

(5.4.1) $$d < \max\{2\sqrt{s}, (p-1)\sqrt{s}\},$$

then G is elementary abelian.

Jungnickel [22] proved that any affine plane A extending a translation net N of small deficiency has to be a translation plane (with the same translation group). We now state an improvement of this result.

Theorem 5.5. *Let N be a translation net of small deficiency $d = s + 1 - r$ with translation group G, and assume that N has order $s \neq 2, 4$. Then the number of transversals of N is a multiple of s, say ts, where $t \leqslant d$; adjoining all these transversals to N results in a transversal-free $(s, r + t)$-net $E = E(D)$ which is actually a translation net with the same translation group G. In particular, the partial congruence partition describing E is obtained from that describing N by adding t further components. Moreover, the automorphism group of N is the stabilizer of N in the automorphism group of E.*

In the case of critical deficiency, the situation is more involved.

Theorem 5.6. *Let N be a translation net of order $s = m^2$, where $m \neq 2$, with critical deficiency $d = m + 1$ belonging to the partial congruence partition $\mathbb{H}$ in the group G. Any transversal of N through 0 is a subgroup of G and can be adjoined as a further component to $\mathbb{H}$. If E is a net extending N, then E is either a translation net with translation group G, or E is a transversal-free net with deficiency m. In the latter case N also possesses at least two distinct extensions to a translation net of deficiency m.*

Any partial congruence partition in an elementary abelian group can be considered as a *partial t-spread* in a suitable projective space $\mathrm{PG}(2t + 1, p)$ (see, e.g., [4] or [24]). If the partial t-spread is in fact maximal, we can apply Theorems 5.5 and 5.6.

Corollary 5.7. *Let $\mathbb{F}$ be a maximal partial t-spread in $\mathrm{PG}(2t + 1, p)$, where p is a prime, and let N be the corresponding translation net. If $\mathbb{F}$ has small or critical deficiency, then N is transversal-free.*

All that remains is to find the required maximal partial t-spreads to be used in Corollary 5.7. While some families of such spreads were known (see the discussion in Jungnickel [24]), we now give a general construction method motivated by ideas of Bruen [8] and Beutelspacher [5]. For simplicity, we only consider the case $t = 1$ in detail; see Hirschfeld [20] for the required geometric background.

Proposition 5.8. *Let S be a* regular spread *in* $\mathrm{PG}(3, q)$, *where $q \geqslant 4$, and let R_0 and R_1 be two* reguli *in S which intersect in exactly two lines, say L and M. Let $\mathbb{H}$ denote the partial spread of deficiency $2q$ obtained by omitting the $2q$ lines in $R_0 \cup R_1$, and let L be a set of lines in* $\mathrm{PG}(3, q)$ *satisfying the following conditions:*

(5.8.1) *L is contained in the union $R_0' \cup R_1'$ of the opposite reguli of R_0 and R_1.*

(5.8.2) *Each point in $L \cup M$ is on at most one line of L.*

(5.8.3) *L contains at least one line from each of R_0' and R_1'.*

(5.8.4) *For each line $G \in (R_0' \cup R_1') - L$, at least one of the two points of intersection of G with $L \cup M$ is on a line of L.*

Then the set $\mathbb{F} = \mathbb{H} \cup \boldsymbol{L}$ *is a maximal partial spread in* PG$(3, q)$ *with deficiency* $d = 2q - |\boldsymbol{L}|$. *Moreover, any line not in* $\mathbb{F}$ *either lies entirely in* $P(\boldsymbol{R}_0) \cup P(\boldsymbol{R}_1)$ (*where* $P(\boldsymbol{X})$ *denotes the set of points covered by the lines in a set of lines* $\boldsymbol{X}$) *or contains at most* 4 *points not in* $P(\mathbb{F})$.

One next shows that any maximal partial spread of PG$(3, q)$ which is constructed by the method of Proposition 5.8 remains maximal when considered as a partial t-spread or the prime subfield GF(p) of GF(q) (for the appropriate value of t). Examples can then be provided by comparatively simple direct computations. We just give the final result on transversal-free nets arising in this way.

It is obvious from (5.8.2) that any set $\boldsymbol{L}$ satisfying the conditions in Proposition 5.8 contains at most $q + 1$ lines. Thus the maximal partial spread belonging to $\boldsymbol{L}$ has deficiency at least $q - 1$, a bound which can be achieved by Theorem 5.9.

Theorem 5.9. *There exist transversal-free translation nets of order* $s = q^2$ *and deficiencies* $d = q - 1$, q *and* $q + 1$ *whenever* q *is a power of a prime* $\geqslant 5$. *Hence there exists a maximal set of* k *MOLS of order* q^2 *for* $k \in \{q^2 - q - 2,\ q^2 - q - 1,\ q^2 - q\}$ *whenever* q *is a power of a prime* $\geqslant 5$.

Some examples can also be constructed for $p = 2$ and $p = 3$, cf. Jungnickel [25]. In general, it is an open problem to decide when a maximal PCP gives rise to a transversal-free net (even if it belongs to a maximal partial t-spread). In this connection, the following result is somewhat surprising.

Theorem 5.10. *Every maximal partial congruence partition in* $G = \mathbb{Z}_q \times \mathbb{Z}_q$ (q *a prime power*) *defines a translation net with translation group* G *which is actually transversal-free. Moreover, all the maximal PCPs in this group can be described.*

6. Codes of translation nets

Let $\boldsymbol{N}$ be a net of order s and degree r. The GF(p)-*code* $C(\boldsymbol{N})$ is the GF(p)-span of the incidence vectors of the lines of $\boldsymbol{N}$; the dimension of $C(\boldsymbol{N})$ is called the p-*rank* of $\boldsymbol{N}$. Moorehouse [30] has proposed the following conjecture.

Moorehouse's Conjecture 6.1. Let $\boldsymbol{N}$ be an (s, r)-net and $\boldsymbol{N}'$ an $(s, r - 1)$-subnet of $\boldsymbol{N}$. If p is a prime dividing s for which p^2 does not divide s, one has

(6.1.1) $\operatorname{rank}_p(\boldsymbol{N}) - \operatorname{rank}_p(\boldsymbol{N}') \geqslant s - r + 1$.

Moorehouse tested his conjecture for numerous small examples and proved its validity for $r = 3$ (and arbitrary $\boldsymbol{N}, \boldsymbol{N}'$) and for subnets of AG$(2, p)$ (and arbitrary r). In the latter case, one has in fact equality in (6.1.1); this strengthens the well-known fact that AG$(2, p)$ has p-rank $p(p+1)/2$, cf. Lander [26]. The importance (and difficulty) of Conjecture 6.1 can be seen from the following result.

Theorem 6.2 (Moorehouse [30]). *Assume the existence of an affine plane of order s, where either $s \equiv 2 \pmod 4$ or s is square-free. If Conjecture* 6.1 *holds for s, then s is a prime and the only plane of order s is* AG$(2, s)$.

Thus Conjecture 6.1 includes the well-known conjecture that AG$(2, p)$ is the only plane of order p. For translation nets, Moorehouse [31] proved the following result.

Theorem 6.3. *Conjecture* 6.1 *holds for translation nets with abelian translation group.*

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